On *bi*soft bitopological spaces

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Abstract

In this paper, a new concept of *bi*-soft bitopological space is introduced. Then some basic notions, including the *bi*-soft open sets, *bi*-soft closed sets, *bi*-soft closure, *bi*-soft interior points, *bi*-soft neighborhood of a point and *bi*-soft separation axioms, in the above space setting, are presented. Furthermore, it is investigeted that, as spaces with two topologies each, quasi metric and asymmetric *bi*-normed spaces endowered with a directed *bi*-gragh are viewed as *bi*-soft bitopological spaces.

Keywords: *bi*-soft bitopological space; quasi *bi*-metric space; asymmetric *bi*-norm space; and directed *bi*-graph.

1 Introduction

There exist a number of theories, such as, fuzzy sets theory [1], intuitionistic fuzzy sets theory [2], vague sets theory, interval mathematics theory [4, 5] and rough sets theory [6] which can be considered as tools for dealing with uncertainties, for example, of natural environmental phenomena; or of human knowledge about the real world; or to the limitations of the means used to measure objects; or in the boundary between states; or between urban and rural areas; or of the exact growth rate of population in a country's rural area; or of making decisions in a machine based environment using database information [7]. But all these theories have their own difficulties. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theories [8]. In [8], the concept of soft set theory as a new mathematical tool which is free from the problems mentioned above was established. Researches on soft set theory and its applications in various fields are progressing rapidly, see for example

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[9, 11, 12, 13, 14, 15, 16, 18, 19, 20].

Let X be an initial universe and E a set of parameters. Let $P(X)$ denote the power set of X and A a non-empty subset of E. A pair $(F, A) = \{(x, F(x)) : x \in A, F(x) \in P(X)\}$ is called a soft set over X, where F is a mapping given by $F : A \to P(X)$.
The relative complement of a soft set (F, A) denoted by $(F, A)'$ is defined by $(F, A)' = (F', A)$ where $F' : A \to P(X)$ is a mapping given by $F'(x) = X - F(x)$ for all $x \in A$. Let (F, A) and (G, B) be soft set over a common universe X, we say that (F, A) is a soft subset of (G, B) (denoted by $(F, A)\overline{\subset}(G, B)$) if $A \subset B$ and for all $x \in A$, $F(x)$ and $G(x)$ are identical approximations. Let $x \in X$. Then $x \in (F, E)$ if $x \in F(y)$ for all $y \in E$. Furthermore, for any $x \in X$, $x \notin (F, E)$, if $x \notin F(y)$ for some $y \in E$. The sub soft set of (F , E) over Y, a non-empty subset of X, is denoted by $({}^{Y}F, E) = (Y, E) \cap (F, E)$, where $Y(x) = Y$ for all $x \in E$.

Let τ be a collections of soft sets over X. Then τ is called a soft topology on X if

1. the empty set $\emptyset$ and a soft set (X, E) belong to τ;
2. the union of any number of soft sets in τ belongs to τ; and
3. the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X. In this case the members (F, E) of τ are said to be soft open sets in X, while a soft set (F, E) over X is a soft closed set in X, if its relative complement $(F, E)'$ belongs to τ. Let (X, τ, E) be a soft topological space over X and Y be a non-empty subset of X. Then $\tau_Y = \{({}^{Y}F, E) : (F, E) \in \tau\}$ is said to be the soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E). The soft closure of (F, E), denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E). Let (G, E) be a soft set over X and $x \in X$. Then x is said to be a soft interior point of (G, E) if there exists a soft open set (F, E) such that $x \in (F, E)\overline{\subset}(G, E)$. Hence, (G, E) is called a soft neighborhood of x.
A soft topological space (X, τ, E) is

1. soft T_0 if for any pair x, y of distinct points in X , at least one of them has a soft neighborhood not containing the other;
2. soft T_1 if for any pair x, y of distinct points in X , each of them has a soft neighborhood not containing the other;
3. soft Hausdorff or soft T_2 if for any pair x, y of distinct points in X , there exist soft neighborhoods (F, E) of x and (G, E) of y such that $(F, E) \cap (G, E) = \emptyset$;
4. soft regular if for each $x \in X$ and each soft closed subset (F, E) of X, not containing x, there are disjoint soft open sets (G, E), (H, E) of τ such

that $x \in (G, E)$ and $(F, E)\tilde{\subset}(H, E)$. If X is both soft regular and soft T_1, then it is called soft T_3.

5. soft normal if for any two disjoint soft closed subsets (F, E) and (G, E) of X, there are two disjoint soft open sets (F_1, E) and (G_1, E) of τ such that $(F, E)\tilde{\subset}(F_1, E)$ and $(G, E)\tilde{\subset}(G_1, E)$. If X is both soft normal and soft T_1, then it is called soft T_4.

A quasi-metric on a set X is a non-negative real-valued function $p(\cdot, \cdot)$ on the product $X \times X$ such that

1. $p(x, x) = 0 \ \ \forall x \in X$;
2. $p(x, z) \leq p(x, y) + p(y, z) \ \ \forall x, y, z \in X$;
3. $p(x, y) = 0 \Leftrightarrow x = y \ \ \forall x, y \in X$.

Now we say that $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$, defined by $q(x, y) = p(y, x)$ are conjugate, and the set X together with this structure, denoted by (X, p, q), is called a quasi-bi-metric space. As in the classical case, the collection of all open p-spheres forms a base for a topology. The topology determined in this way by $p(\cdot, \cdot)$ will be denoted by τ_p and will be called the quasi-metric topology of $p(\cdot, \cdot)$. Similarly, $q(\cdot, \cdot)$ determines a topology μ_q, which will be called the conjugate quasi-metric topology of $q(\cdot, \cdot)$. Thus the natural topological structure associated with a quasi-metric and conjugate quasi-metric on a set X is that of the set X with two topologies.
An asymmetric norm is a positive sublinear functional $\|\cdot|$ on a real vector space X which satisfies $\|x| = \| - x| = 0$ implies $x = 0$. The conjugate asymmetric norm $|\cdot\|$ is given by $|x\| := \| - x|$, whenever $x \in X$ is already the pre-image of $\|\cdot|$. X together with $\|\cdot|$ and $|\cdot\|$ forms a real asymmetric binormed space. Let X be a real vector space. A quasi bi-metric on X is induced by an asymmetric binorm on X in the form: $p(x, y) = \|x - y|$ and $q(x, y) = |x - y\|$. The balls with respect to $\|\cdot|$, denoted by $B^{\|\cdot|}$, are called forward balls and the topology $\tau_{\|\cdot|}$ generated by $\|\cdot|$ is called the forward topology, while the balls with respect to $|\cdot\|$, given by $B^{|\cdot\|}$, are called backward balls and the topology $\tau_{|\cdot\|}$ generated by $|\cdot\|$ the backward topology. The details of the last two distance maps are discussed in [3].
In general, a set X endowed with two toplogies (say) τ and μ is called bitopological space. The separation axioms specific to bitopological spaces were first studied by Kelly [21], and they are defined, for $\wedge$ and $\vee$ denote "and" and "or", respectively, as follow:

1. Pairwise T_0 spaces: for $x, y \in X$ distinct, there exists $U \subset \tau$ and $V \subset \mu$ such that

$$[((x \in U) \wedge (y \notin U)) \vee ((y \in U) \wedge (x \notin U))] \\ \wedge [((x \in V) \wedge (y \notin V)) \vee ((y \in V) \wedge (x \notin V))].$$

2. Pairwise T_1 spaces: for $x, y \in X$ distinct, there exists dintincts $U_1, U_2 \subset \tau$ and $V_1, V_2 \subset \mu$ such that

$$[((x \in U_1) \wedge (y \notin U_1)) \wedge ((y \in V_1) \wedge (x \notin V_1))] \\ \wedge [((y \in U_2) \wedge (x \notin U_2)) \wedge ((x \in V_2) \wedge (y \notin V_2))].$$

3. Pairwise Hausedorff spaces: for $x, y \in X$ distinct, there exists disjoints $U_1, U_2 \subset \tau$ and $V_1, V_2 \subset \mu$ such that

$$[((x \in U_1) \wedge (y \notin U_1)) \wedge ((y \in V_1) \wedge (x \notin V_1))] \\ \wedge [((y \in U_2) \wedge (x \notin U_2)) \wedge ((x \in V_2) \wedge (y \notin V_2))].$$

4. Pairwise regular spaces: for $x \in X$, $S \subset X$ closed; x and S are disjoint, then there exists disjoints $U_1, U_2 \subset \tau$ and $V_1, V_2 \subset \mu$ such that

$$[((x \in U_1) \wedge (S \notin U_1)) \wedge ((S \in V_1) \wedge (x \notin V_1))] \\ \wedge [((S \in U_2) \wedge (x \notin U_2)) \wedge ((x \in V_2) \wedge (S \notin V_2))].$$

5. Pairwise normal spaces: for $S_1, S_2 \in X$ closed and $S_1 \cap S_2 = \emptyset$, then there exists disjoints $U_1, U_2 \subset \tau$ and $V_1, V_2 \subset \mu$ such that

$$[((S_1 \in U_1) \wedge (S_2 \notin U_1)) \wedge ((S_2 \in V_1) \wedge (S_1 \notin V_1))] \\ \wedge [((S_2 \in U_2) \wedge (S_1 \notin U_2)) \wedge ((S_1 \in V_2) \wedge (S_2 \notin V_2))].$$

In [22, 23], a directed graph G on an arbitrary set X is an ordered pair $(V(G), E(G)) = \{((x, y), (x, y)_R) : (x, y) \in X \times X \text{ and } (x, y)_R \in V(G) \times V(G)\}$, such that $V(G) \subset X$ is a nonempty set called a set of vertices, and $E(G) = \{(x, y)_R \in V(G) \times V(G) : xRy$ (i.e., y is related to y)$\}$ is a binary relation R on V whose elements are called edges. Converse directed graph G^{-1} is $(V(G^{-1}), E(G^{-1})) := \{((x, y), (y, x)_R) : ((x, y), (x, y)_R) \in (V(G), E(G))\}$. For example, consider the set of natural numbers $K := \{1, 2, 3, 4, 5, 6\}$ and let $D = \{(x, y) : x, y \in K\}$ be the set of all possible combinations (x, y) of K members taking two at a time, then the pair (K, D) define a directed graph G on K with $V(G) = K$ and $E(G) = D$, while (K, D') define its converse directed graph G^{-1}, where $D' := \{(y, x) : (x, y) \in D\}$. So, as the directed graph is not in general symmetry, the natural distance function between its vertices assigned to the corresponding edges is quasi-metric. Similarly, the converse directed graph will associate with the conjugate quasi-metric. Hence $(V(G), E(G), E(G^{-1}))$ can be seen as a directed *bi*-graph and so the bitopological structure $(X, \tau_p, \mu_q, G, G^{-1})$ will be called quasi-*bi*-metric bitopological space with a directed *bi*-graph. In a similar way, we get another bitopological structure $(X, \tau_{\|\cdot|}, \mu_{|\cdot\|}, G, G^{-1})$ called an

asymmetric *bi*-normed bitopological space with a directed *bi*-graph.
For example:

1. On the set of real numbers $\mathbb{R}$, if $\rho \in \{p, q\}$, $k > 0$ and

$$\rho(x,y) := \begin{cases} p(x,y) = y - x, \text{ for } x \leq y \\ q(x,y) = k(x-y), \text{ for } y \leq x, \end{cases}$$

   then, for $V(G) = \mathbb{R}$, $E(G) = \{(x,y) : x \leq y\}$, $E(G^{-1}) = \{(y,x) : (x,y) \in E(G)\}$ and $(\mathbb{R}, E(G), E(G^{-1}))$ define a directed *bi*-graph, the space $(\mathbb{R}, p, q, \leq, \geq)$ is a quasi-*bi*-metric space with a directed *bi*-graph but not a metric space with a graph.

2. Let $\eta \in \{\|\cdot|, |\cdot\|\}$. Similarly, on the real line $\mathbb{R}$, if $k > 0$ and

$$\eta(x) = \begin{cases} \|x| = k|x|, x \leq 0 \\ |x\| = |x|, x \geq 0, \end{cases}$$

   then $(\mathbb{R}, \|\cdot|, |\cdot\|, \leq, \geq)$ is an asymmetric *bi*-normed space with a directed *bi*-graph but not a normed space with a graph.

The main purpose of this paper is to introduce and investigate some new definitions and properties which extend the notions of quasi-*bi*-metric and asymmetric *bi*-normed bitopological spaces with a directed *bi*-graph to a new, as far as we know, and more general bitopological structure called *bi*-soft bitopological space.

2 *bi*Soft sets

Definition 2.1 *Let X be an initial universe and E be a set of parameters. Let $P(X)$ be the power set of X and A be a non-empty subset of E. Let F be a bifunction given by $F : A \times A \to P(X)$. A pair*

$$(F, A) = \{((x,y), F(x,y)) : (x,y) \in A \times A \text{ and } F(x,y) \in P(X)\}$$

is called a soft set over X. For $\overline{F}$ (a converse of F) defined by $\overline{F}(x,y) := F(y,x)$, a triplet $(F, \overline{F}, A)$ is what we refer to as a bi-soft set over X. Where, a pair

$$(\overline{F}, A) = \{((x,y), F(y,x)) : ((x,y), F(x,y)) \in (F, A)\}$$

is called a converse soft set over X.

Remark 2.1 *For $(x, y) \in A \times A$, $F(x, y)$ ($\overline{F}(x, y)$) may be regarded as the set of (x, y)-approximate elements ((y, x)-approximate elements) of the soft set (F, A) (respectively, converse soft set $(\overline{F}, A)$).*

To illustrate the above definition, let us consider the following example.

Example 2.1 *A bi-soft set $(F, \overline{F}, E)$ describes the cultural differences which a couple might choose to use.*

X - is the set of cultures under consideration.
$X :=$ {'Sex', 'Marraige', 'Love', 'Sex before marraige', 'Sex before love', 'Marraige before sex', 'Marraige before love', 'Love before marraige', 'Love before sex'}.

E - is the set of parameters.
$E := \{Sex, Marraige, Love\}$.

In this case, to define a bi-soft set means to point out cultures for the couple, such as:
$F(Sex, Sex) =$ {'Sex'};
$F(Marraige, Marraige) =$ {'Marraige'};
$F(Love, Love) =$ {'Love'};
$F(Sex, Marraige) =$ {'Sex before marraige'};
$\overline{F}(Sex, Marraige) =$ {'Marraige before sex'};
$F(Sex, Love) =$ {'Sex before love'};
$\overline{F}(Sex, Love) =$ {'Love before Sex'};
$F(Marraige, Love) =$ {'Marraige before love'}; and
$\overline{F}(Marraige, Love) =$ {'Love before Marraige'}.

Definition 2.2 *Let $(F, \overline{F}, A)$ and $(G, \overline{G}, B)$ be bi-soft set over a common universe X, we say that $(F, \overline{F}, A)$ is a bi-soft subset of $(G, \overline{G}, B)$ (denoted by $(F, \overline{F}, A)\tilde{\subset}(G, \overline{G}, B)$) if $A \subset B$ and for all $(x, y) \in A \times A$, $F(x, y)$ is identical approximation to $G(x, y)$ and $\overline{F}(x, y)$ is identical approximation to $\overline{G}(x, y)$.*

Definition 2.3 *Given two bi-soft sets $(F, \overline{F}, A)$ and $(G, \overline{G}, B)$ over a common universe X, we say that $(F, \overline{F}, A)$ is a bi-soft subset of $(G, \overline{G}, B)$, denoted by*

$(F, \overline{F}, A)\tilde{\subset}(G, \overline{G}, B)$, *if*

1. $A \subset B$ *and*
2. $\forall (x, y) \in A \times A$, $F(x, y)$ *is identical approximation to* $G(x, y)$ *and* $\overline{F}(x, y)$ *is identical approximation to* $\overline{G}(x, y)$.

Definition 2.4 *Let* $E = \{x_1, x_2, x_3, ..., x_n\}$ *be a set of parameters. The complement of* E, *denoted by* E^c *is defined by* $E^c = \{x_1^c, x_2^c, x_3^c, ..., x_n^c\}$ *where* x_i^c *are* x_i *complemets* $\forall i$.

Definition 2.5 *The complement of* $(F, \overline{F}, A)$, *denoted by* $(F, \overline{F}, A)^c$, *is given by* $(F, \overline{F}, A)^c = (F^c, \overline{F}^c, A^c)$ *where* $F^c : A^c \times A^c \to P(X)$ *is defined by* $F^c(x^c, y^c) = X - F(x, y)$, $\forall (x^c, y^c) \in A^c \times A^c$. *For the converse* $\overline{F}^c$, *we have* $\overline{F}^c(x^c, y^c) = X - F(y, x)$ *whenever* $F(x, y) \in P(X)$.

Definition 2.6 *Let* $(F, \overline{F}, E)$ *and* $(G, \overline{G}, E)$ *be two bi-soft sets over* X. *Then* $(F, \overline{F}, E)\backslash(G, \overline{G}, E)$, *denoted by* $(H, \overline{H}, E)$, *is defined as* $H(x, y) = F(x, y)\backslash G(x, y)$ *and* $\overline{H}(x, y) = \overline{F}(x, y)\backslash\overline{G}(x, y)$ *for all* $(x, y) \in E$.

Definition 2.7 *A bi-soft set* $(F, \overline{F}, A)$ *over* X *is said to be a null bi-soft set, denoted by* $\overline{\emptyset}$, *if for all* $(x, y) \in A \times A$, $F(x, y) = \overline{F}(x, y) = \emptyset$.

Definition 2.8 *Let* $x \in X$. *Then* $x \in (F, \overline{F}, E)$ *if* $x \in F(y, z) \cup \overline{F}(y, z)$ *for all* $(y, z) \in E \times E$. *Furthermore, for any* $x \in X$, $x \notin (F, \overline{F}, E)$, *if* $x \notin F(y, z) \cup \overline{F}(y, z)$ *for some* $(y, z) \in E \times E$.

Definition 2.9 *Let* Y *be a non-empty subset of* X, *then* $\tilde{\tilde{Y}}$ *denotes the bi-soft set* $(Y, \overline{Y}, E)$ *over* X *for which* $Y(x, y) = Y$ *and* $\overline{Y}(x, y) = \overline{Y}$, *for all* $(x, y) \in E \times E$. *In particular,* $(X, \overline{X}, E)$ *will be denoted by* $\tilde{\tilde{X}}$.

Definition 2.10 *The sub bi-soft set of* $(F, \overline{F}, E)$ *over* Y *a non-empty subset of* X *denoted by* $({}^YF, {}^Y\overline{F}, E) = \tilde{\tilde{Y}} \cap (F, \overline{F}, E)$, *where* ${}^YF(x, y) = Y \cap F(x, y)$ *and* ${}^Y\overline{F}(x, y) = \overline{Y} \cap \overline{F}(x, y)$, *for all* $(x, y) \in E \times E$.

Definition 2.11 *A bi-soft set* $(F, \overline{F}, A)$ *over* X *is said to be an absolute bi-soft set, denoted by* $\overline{A}$, *if for all* $(x, y) \in A \times A$, $F(x, y) \cup \overline{F}(x, y) = X$. *Furthermore,* $\overline{A}^c = \overline{\emptyset}$ *and* $\overline{\emptyset}^c = \overline{A}$.

Definition 2.12 *Let* $(F, \overline{F}, A)$ *and* $(G, \overline{G}, B)$ *be two bi-soft sets over* X. *Then*

$(H, \overline{H}, A \cup B)$ *is said to be their union, denoted by* $(F, \overline{F}, A) \overline{\cup} (G, \overline{G}, B) = (H, \overline{H}, A \cup B)$*, if for all* $(x, y) \in C \times C = (A \cup B) \times (A \cup B)$*,*

$$H(x,y) = \begin{cases} F(x,y), \ if \ (x,y) \in (A-B) \times (A-B) \\ G(x,y), \ if \ (x,y) \in (B-A) \times (B-A) \\ F(x,y) \cup G(x,y), \ if \ (x,y) \in (A \cap B) \times (A \cap B) \end{cases}$$

and

$$\overline{H}(x,y) = \begin{cases} \overline{F}(x,y), \ if \ (x,y) \in (A-B) \times (A-B) \\ \overline{G}(x,y), \ if \ (x,y) \in (B-A) \times (B-A) \\ \overline{F}(x,y) \cup \overline{G}(x,y), \ if \ (x,y) \in (A \cap B) \times (A \cap B). \end{cases}$$

Example 2.2 *From Example 2.1, consider* $A = \{Sex, Marraige\}$ *and* $B = \{Sex, Love\}$ *such that*
$F(Sex, Sex) = \{$*'Sex'*$\}$*;*
$F(Marraige, Marraige) = \{$*'Marraige'*$\}$*;*
$F(Sex, Marraige) = \{$*'Sex before marraige'*$\}$*;*
$\overline{F}(Sex, Marraige) = \{$*'Marraige before sex'*$\}$*;*
$G(Sex, Sex) = \{$*'Sex'*$\}$*;*
$G(Love, Love) = \{$*'Love'*$\}$*;*
$G(Sex, Love) = \{$*'Sex before love'*$\}$*;*
$\overline{G}(Sex, Love) = \{$*'Love before Sex'*$\}$*. Hence, we have*
$H(Sex, Sex) = \{$*'Sex'*$\}$*;*
H(Marraige, Marraige)={'Marraige'};
H(Love, Love)={'Love'};
$H(Sex, Marraige) = \overline{\emptyset}$*;*
$\overline{H}(Sex, Marraige) = \overline{\emptyset}$*;*
$H(Sex, Love) = \overline{\emptyset}$*;*
$\overline{H}(Sex, Love) = \overline{\emptyset}$*;*
$H(Marraige, Love) = \overline{\emptyset}$*;*
$\overline{H}(Marraige, Love) = \overline{\emptyset}$.

Definition 2.13 *The intersection* $(H, \overline{H}, A \cap B)$ *of two bi-soft sets* $(F, \overline{F}, A)$

and $(G, \overline{G}, B)$ *over a common univserse* X*, denoted by* $(F, \overline{F}, A)\overline{\cap}(G, \tilde{G}, B)$ *is defined as* $H(x, y) = F(x, y) \cap G(x, y)$ *and* $\overline{H}(x, y) = \overline{F}(x, y) \cap \overline{G}(x, y)$*, for all* $(x, y) \in A \cap B \times A \cap B$*.*

Example 2.3 *From Example 2.2, intersection of two bi-soft sets* $(F, \overline{F}, A)$ *and* $(G, \overline{G}, B)$ *is the bi-soft set* $(H, \overline{H}, A \cap B)$ *where* $A \cap B = \{$*'Sex'*$\}$ *and* $H(Sex, Sex) = \{$*'Sex'*$\}$*.*

3 Operations with *bi*Soft sets

Definition 3.1 *Let* $*$ *be a binary operation for subsets of the set* X *and let* $(F, \overline{F}, A)$ *and* $(G, \overline{G}, B)$ *be biSoft sets over* X*. Then the operation* $*$ *is defined as follows:*

$$(F, \overline{F}, A) * (G, \overline{G}, B) = (H, \overline{H}, A \times B),$$

where $H((x, y), (w, z)) = F(x, y) * G(w, z)$*,* $\overline{H}((y, x), (z, w)) = \overline{G}(w, z) * \overline{F}(x, y)$*,* $(x, y) \in A \times A$*,* $(w, z) \in B \times B$*, and* $((x, y), (w, z)) \in (A \times A) \times (B \times B)$*.*

Example 3.1 *From Example 2.1, the intersection of the bi-soft sets with itself yields the bi-soft set with more detailed description. The resulting bi-soft sets points out the cultures which are 'Islamic'* (I) *and 'Unislamic'* (U)*. These are,*

$I((Marraige, Marraige), (Sex, Sex), (Love, Love), (Marraige, Sex),$
$(Love, Marraige), (Love, Sex))$
$=$ {*'Marraige', 'Sex', 'Love', 'Marraige before sex', 'Love before marraige', 'Love before sex'*}*.*

$U((Marraige, Sex), (Love, Marraige), (Love, Sex))$
$=$ {*'Sex before marraige', 'Marraige before love', 'Sex before love'*}*.*

3.1 *bi*Soft bitopology and *bi*Soft bitopological spaces

Definition 3.2 *Let* τ *and* μ *be two collections of soft sets and converse soft sets over* X*, respectively. Then* τ (μ) *is called a soft topology (converse soft topology) on* X *if*

1. *the empty set $\emptyset$ and a soft set (X, E) belong to τ;*

2. *the union of any number of soft sets in τ belongs to τ;*

3. *the intersection of any two soft sets in τ belongs to τ*

(respectively,

1. *the empty set $\emptyset$ and a converse soft set $(\overline{X}, E)$ belong to μ;*

2. *the union of any number of converse soft sets in μ belongs to μ;*

3. *the intersection of any two converse soft sets in μ belongs to μ).*

The set of bi-soft sets $(X, \overline{X}, E) = (X, E)\tilde{\cup}(\overline{X}, E)$, for all $(X, E) \in \tau$ and $(\overline{X}, E) \in \mu$, is called bi-soft bitopology on X.

Definition 3.3 *Let (X, τ, μ, E) be a soft bitopological space over X and Y be a non-empty subset of X. Then $\tau_Y = \{({}^Y F, E) : (F, E) \in \tau\}$ ($\mu_Y = \{({}^Y \overline{F}, E) : (\overline{F}, E) \in \mu\}$) is said to be the soft relative topology (converse soft relative topology) on Y and (Y, τ_Y, E) ((Y, μ_Y, E)) is called a soft subspace of (X, τ, E) (respectively, converse soft subspace of (X, μ, E)). Therefore, (Y, τ_Y, μ_Y, E) is called a bi-soft subspace of (X, τ, μ, E).*

Definition 3.4 *(X, τ, μ, E) is said to be a bi-soft bitopological space and the members of τ (μ) are called soft open sets (respectively, converse soft open sets) in X. Therefore, members of the two topologies all together they form bi-soft open sets.*

Definition 3.5 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $(F, \overline{F}, E)$ be a bi-soft set over X. Then the bi-soft closure of $(F, \overline{F}, E)$, denoted by $\overline{(F, \overline{F}, E)}$ is the intersection of all bi-soft closed super sets of $(F, \overline{F}, E)$.*

Definition 3.6 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $(G, \overline{G}, E)$ be a bi-soft set over X. Then $x \in X$ is said to be a bi-soft interior point of $(G, \overline{G}, E)$ if there exists a bi-soft open set $(F, \overline{F}, E)$ such that $x \in (F, \overline{F}, E)\tilde{\subset}(G, \overline{G}, E)$.*

Definition 3.7 *Let (X, τ, μ, E) be a bi-soft bitopological space over X, $(G, \overline{G}, E)$ be a bi-soft set over X and $x \in X$. Then $(G, \overline{G}, E)$ is said to be a bi-soft neighborhood of x if there exists a bi-soft open set $(F, \overline{F}, E)$ such that $x \in$*

$(F, \overline{F}, E)\overline{\subset}(G, \overline{G}, E)$.

3.2 *bi*-soft separation axioms

Definition 3.8 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $x, y \in X$ such that $x \neq y$. If there exists bi-soft open sets $(F, \overline{F}, E) \subset \tau$ and $(G, \overline{G}, E) \subset \mu$ such that*

$$\begin{aligned}[((x \in (F, \overline{F}, E)) \wedge (y \notin (F, \overline{F}, E))) \vee ((y \in (F, \overline{F}, E)) \wedge (x \notin (F, \overline{F}, E)))] \\ \wedge [((x \in (G, \overline{G}, E)) \wedge (y \notin (G, \overline{G}, E))) \\ \vee ((y \in (G, \overline{G}, E)) \wedge (x \notin (G, \overline{G}, E)))],\end{aligned}$$

then (X, τ, μ, E) is called a bi-soft pairwise T_0-space.

Definition 3.9 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $x, y \in X$ such that $x \neq y$. If there exists dintincts bi-soft open sets $(F_1, \overline{F_1}, E), (F_2, \overline{F_2}, E) \subset \tau$ and $(G_1, \overline{G_1}, E), (G_2, \overline{G_2}, E) \subset \mu$ such that*

$$\begin{aligned}[((x \in (F_1, \overline{F_1}, E)) \wedge (y \notin (F_1, \overline{F_1}, E))) \wedge ((y \in (G_1, \overline{G_1}, E)) \wedge (x \notin (G_1, \overline{G_1}, E)))] \\ \wedge [((y \in (F_2, \overline{F_2}, E)) \wedge (x \notin (F_2, \overline{F_2}, E))) \\ \wedge ((x \in (G_2, \overline{G_2}, E)) \wedge (y \notin (G_2, \overline{G_2}, E)))],\end{aligned}$$

then (X, τ, μ, E) is called a bi-soft pairwise T_1-space.

Definition 3.10 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $x, y \in X$ such that $x \neq y$. If there exists disjoints bi-soft open sets $(F_1, \overline{F_1}, E), (F_2, \overline{F_2}, E) \subset \tau$ and $(G_1, \overline{G_1}, E), (G_2, \overline{G_2}, E) \subset \mu$ such that*

$$\begin{aligned}[((x \in (F_1, \overline{F_1}, E)) \wedge (y \notin (F_1, \overline{F_1}, E))) \wedge ((y \in (G_1, \overline{G_1}, E)) \wedge (x \notin (G_1, \overline{G_1}, E)))] \\ \wedge [((y \in (F_2, \overline{F_2}, E)) \wedge (x \notin (F_2, \overline{F_2}, E))) \\ \wedge ((x \in (G_2, \overline{G_2}, E)) \wedge (y \notin (G_2, \overline{G_2}, E)))],\end{aligned}$$

then (X, τ, μ, E) is called a bi-soft pairwise T_2-space.

Definition 3.11 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $x, (S, \overline{S}, E) \in X$ such that $(S, \overline{S}, E)$ is closed and $x \notin (S, \overline{S}, E)$. If there exists disjoints $(F_1, \overline{F_1}, E), (F_2, \overline{F_2}, E) \subset \tau$ and $(G_1, \overline{G_1}, E), (G_2, \overline{G_2}, E) \subset \mu$ such that*

$$\begin{aligned}[((x \in (F_1,\overline{F_1}, E)) \wedge ((S, \overline{S}, E) \not\subseteq (F_1, \overline{F_1}, E))) \wedge (((S, \overline{S}, E) \subseteq (G_1, \overline{G_1}, E)) \\ \wedge (x \notin (G_1, \overline{G_1}, E)))] \wedge [(((S, \overline{S}, E) \subseteq (F_2, \overline{F_2}, E)) \wedge (x \notin (F_2, \overline{F_2}, E))) \\ \wedge ((x \in (G_2, \overline{G_2}, E)) \wedge ((S, \overline{S}, E) \not\subseteq (G_2, \overline{G_2}, E)))],\end{aligned}$$

then (X, τ, μ, E) is called a bi-soft pairwise regular space. If (X, τ, μ, E) is both bi-soft pairwise regular and T_1 space then it is called a T_3-space.

Definition 3.12 *Let (X, τ, μ, E) be a bi-soft bitopological space over X and $S_1 := (S_1, \overline{S_1}, E)$ and $S_2 := (S_2, \overline{S_2}, E)$ such that $S_1, S_2 \subset X$ are closed and disjoints. If there exists disjoints $(F_1, \overline{F_1}, E), (F_2, \overline{F_2}, E) \subset \tau$ and $(G_1, \overline{G_1}, E), (G_2, \overline{G_2}, E) \subset \mu$ such that*

$$\begin{aligned}[((S_1 \subseteq (F_1, \overline{F_1}, E)) \wedge (S_2 \not\subseteq (F_1, \overline{F_1}, E))) \wedge ((S_2 \subseteq (G_1, \overline{G_1}, E)) \wedge (S_1 \not\subseteq (G_1, \overline{G_1}, E)))] \\ \wedge [((S_2 \subseteq (F_2, \overline{F_2}, E)) \wedge (S_1 \not\subseteq (F_2, \overline{F_2}, E))) \\ \wedge ((S_1 \subseteq (G_2, \overline{G_2}, E)) \wedge (S_2 \not\subseteq (G_2, \overline{G_2}, E)))],\end{aligned}$$

then (X, τ, μ, E) is called a bi-soft pairwise normal space. If (X, τ, μ, E) is both bi-soft pairwise normal and T_1 space then it is called a T_4-space.

3.3 *bi*Soft bitopology of quasi-*bi*-metric space with a directed *bi*-graph

Definition 3.13 *The soft (converse soft) topology τ_p (μ_q) of quasi-metric (conjugate quasi-metric) space (X, p) ((X, q)) can be defined begining from the family $\mathbb{F}_p[x]$ ($\mathbb{F}_q[y]$) of soft neighborhoods (converse soft neighborhoods) of arbitrary points $x, y \in X$:*

$$\begin{aligned}&(F, E) \in \mathbb{F}_p[x]((\overline{F}, E) \in \mathbb{F}_q[y]) \\ &\iff \exists \epsilon > 0 (\exists r > 0) \ s.t. \ y \in B_p(x, \epsilon) \subset (F, E) (resp., \ x \in B_q(y, r) \subset (\overline{F}, E)).\end{aligned}$$

The two topologies τ_p and μ_q all together forms a bi-soft bitopology of a quasi-bi-metric space with a directed bi-graph.

Remark 3.1 *1. Sometimes, the balls with respect to p are called forwrd balls and the topology τ_p is called the forward topology, while the balls with respect to q are called backward balls and the topology μ_q the backward topology.*

2. To see the rough sketch of the above balls, let $X = \mathbb{R}^2$ be endowed with $\rho_2 \in \{p_2, q_2\}$ and let, for $z \in \{x, y\} \in \mathbb{R}^2$,

$$B_\rho(z) = \begin{cases} x, y \in \mathbb{R}^2 : p_2(x,y) = q_2(x,y) < r, \ if \ \epsilon = r = 1, \\ x, y \in \mathbb{R}^2 : p_2(x,y) < 1, q_2(x,y) < r, \ if \ r \in (0,1), \\ x, y \in \mathbb{R}^2 : p_2(x,y) < 1, q_2(x,y) < r, \ if \ r \in (1,\infty). \end{cases}$$

Then, below we have Figure 1, 2 and 3, respectively.

3. A bi-soft bitopological space (X, τ, μ) is called quasi-bi-metrizable with a directed bi-graph if there exists quasi-metric p and conjugate quasi-metric on X such that $\tau = \tau_p$ and $\mu = \mu_q$.

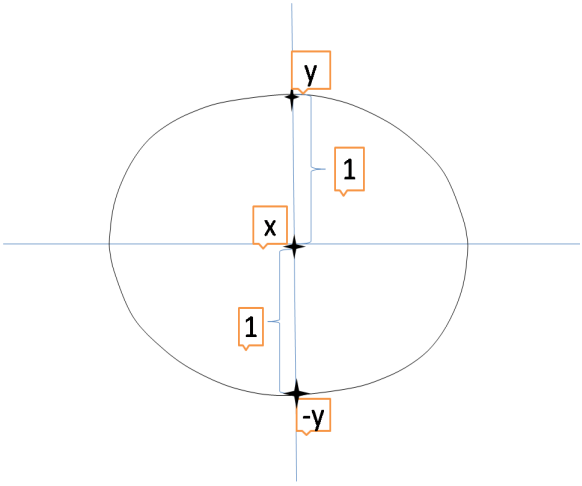

Figure 1: When $\epsilon = r = 1$

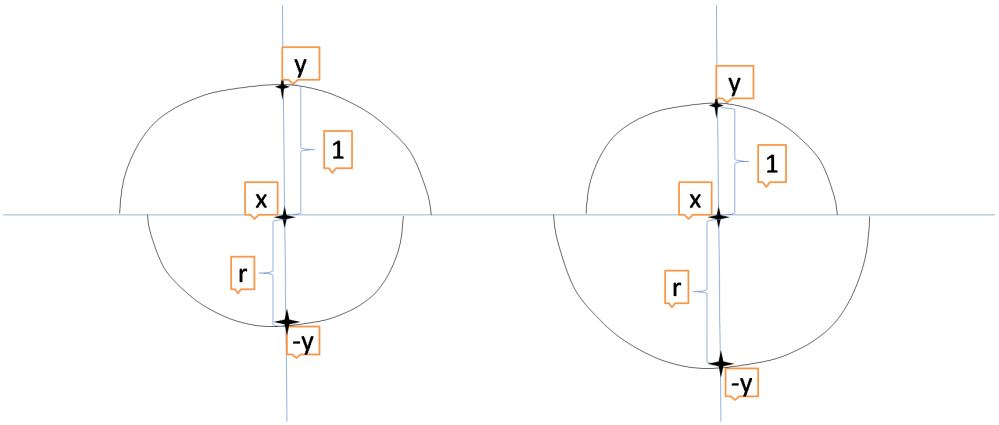

Figure 2: When $\epsilon = 1$, $r < 1$

Figure 3: When $\epsilon = 1$, $r > 1$

3.4 *bi*Soft bitopology of asymmetric *bi*-normed space with a directed *bi*-graph

Definition 3.14 *Let X be real vector space and let the soft (converse soft) topology $\tau_{\|\cdot|}$ ($\mu_{|\cdot\|}$) of asymmetric norm (conjugate asymmetric norm) space $(X, \|\cdot|)$ ($(X, |\cdot\|)$) be defined by the family $\mathbb{F}_{\|\cdot|}[0]$ ($\mathbb{F}_{|\cdot\|}[0]$) of soft neighborhoods (converse soft neighborhoods) of $0 \in X$: for $x \in X$,*

$(F, E) \in \mathbb{F}_{\|\cdot|}[0]((\overline{F}, E) \in \mathbb{F}_{|\cdot\|}[0])$

$\iff \exists \epsilon > 0 (\exists r > 0)$ *s.t.* $x \in B_{\|\cdot|}(0, \epsilon) \subset (F, E) (resp.,\ -x \in B_{|\cdot\|}(0, r) \subset (\overline{F}, E))$.

Then the two topologies $\tau_{\|\cdot|}$ and $\mu_{|\cdot\|}$ all together forms a bi-soft bitopology of asymmetric bi-normed space with a directed bi-graph.

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